MULTIPLE INTEGRALS - TASKS (V PART)

Calculate the surface area

i) If the surface is given by equation z = z (x, y) and if we mark $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$, then:

$$P = \iint_D \sqrt{1 + p^2 + q^2}\, dxdy$$

ii) If the surface is given by parametric equations x = x (u, v) and y = y (u, v), then:

$$P = \iint_D \sqrt{EG - F^2}\, dudv$$ where:

$$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2$$

$$G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2$$

$$F = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}$$

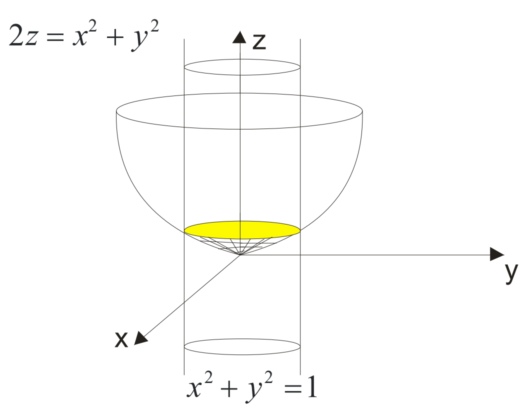
One tip: before studying these ,**remind of the partial derivatives** (you have a file here, on site)

Example 1.

Calculate the area of paraboloid $2z = x^2 + y^2$ that cuts out cylinders $x^2 + y^2 = 1$

Solution:

Picture :



Formula is $\iint\limits_D \sqrt{1+p^2+q^2}\,dxdy$ where z = z(x,y) is paraboloid $z=\frac{x^2}{2}+\frac{y^2}{2}$

and cylinder will give us the limits under which we work ..

$$z=\frac{x^2}{2}+\frac{y^2}{2}$$

$$p=\frac{\partial z}{\partial x}=\frac{2x}{2}=x$$

$$q=\frac{\partial z}{\partial y}=\frac{2y}{2}=y$$

Best to settle on the side (now it is not difficult, but for the second time that we know): $\sqrt{1+p^2+q^2}=\sqrt{1+x^2+y^2}$

$$P=\iint\limits_D \sqrt{1+p^2+q^2}\,dxdy=\iint\limits_D \sqrt{1+x^2+y^2}\,dxdy$$

We said that cylinder $x^2+y^2=1$ determines the borders

We take polar coordinates:

$$\left.\begin{aligned} x&=r\cos\varphi \\ y&=r\sin\varphi \end{aligned}\right\} \to |J|=r \quad \text{then is } 0\le r\le 1 \quad \text{and} \quad 0\le\varphi\le 2\pi$$

$$P=\iint\limits_D \sqrt{1+x^2+y^2}\,dxdy=\underbrace{\boxed{\int_0^{2\pi} d\varphi}}_{\text{This is } 2\pi}\int_0^1 \sqrt{1+r^2}\cdot r\,dr=$$

We'll solve this integral "aside":

$$\int\sqrt{1+r^2}\cdot r\,dr=\left|\begin{aligned}&1+r^2=t^2\\&2rdr=2tdt\\&rdr=tdt\end{aligned}\right|=\int\sqrt{t^2}\cdot t\,dt=\int t^2dt=\frac{t^3}{3}=\frac{\left(\sqrt{1+r^2}\right)^3}{3}$$

Now , we have:

$$P=\iint\limits_D \sqrt{1+x^2+y^2}\,dxdy=\underbrace{\boxed{\int_0^{2\pi} d\varphi}}_{\text{This is } 2\pi}\int_0^1 \sqrt{1+r^2}\cdot r\,dr=2\pi\cdot\left.\frac{\left(\sqrt{1+r^2}\right)^3}{3}\right|_0^1$$

$$=\frac{2\pi}{3}\left(\left(\sqrt{1+1^2}\right)^3-\left(\sqrt{1+0^2}\right)^3\right)=\frac{2\pi}{3}\left(\sqrt{2}^3-1\right)=\frac{2\pi}{3}\left(2\sqrt{2}-1\right)$$

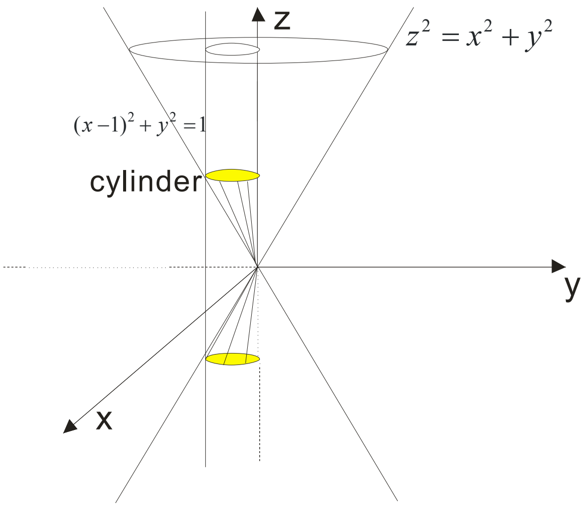
$$\boxed{P=\frac{2\pi}{3}\left(2\sqrt{2}-1\right)}$$

Example 2.

Calculate the area of the cone $z^2 = x^2 + y^2$ which cut cylinder $x^2 + y^2 = 2x$

Solution:

$$x^2 + y^2 - 2x = 0$$
$$x^2 - 2x + 1 - 1 + y^2 = 0$$
$$(x-1)^2 + y^2 = 1$$

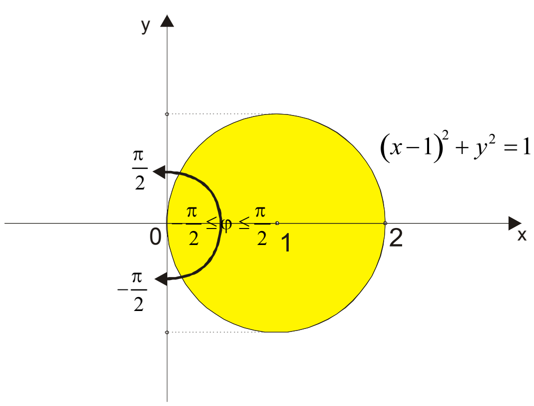


We see that we have two symmetrical surface relative to the plane z = 0.

We'll find one and multiply it by 2.

The cylinder will then give us the borders!

Picture in plane will be:



We take polar coordinates:

$$\left.\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array}\right\} \rightarrow |J| = r \quad \text{then:}$$

$$x^2 + y^2 - 2x = 0$$

$$(r\cos\varphi)^2 + (r\sin\varphi)^2 - 2r\cos\varphi = 0$$

$$r^2(\cos^2\varphi + \sin^2\varphi) = 2r\cos\varphi$$

$$r^2 = 2r\cos\varphi \rightarrow r = 2\cos\varphi$$

So: $0 \le r \le 2\cos\varphi$

From picture we can see that the angle goes from: $-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$

$$z = \pm\sqrt{x^2 + y^2} \rightarrow We \;\; do \quad z = +\sqrt{x^2 + y^2}$$

$$p = \frac{\partial z}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2}} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$q = \frac{\partial z}{\partial y} = \frac{2y}{2\sqrt{x^2 + y^2}} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\sqrt{1 + p^2 + q^2} = \sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2} = \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} =$$

$$= \sqrt{\frac{x^2 + y^2}{x^2 + y^2} + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} == \sqrt{\frac{2\left(x^2 + y^2\right)}{x^2 + y^2}} = \sqrt{2}$$

To find area:

$$P_1 = \iint_D \sqrt{1 + p^2 + q^2}\,dxdy = \iint_D \sqrt{2}dxdy = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} rdr = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{r^2}{2}\right)\Bigg|_0^{2\cos\varphi} d\varphi =$$

$$= \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{4\cos^2\varphi}{2}\right) d\varphi = 2\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\varphi d\varphi = \cancel{2}\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos 2\varphi}{\cancel{2}} d\varphi =$$

$$= \sqrt{2}\left(\varphi + \frac{1}{2}\sin 2\varphi\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \sqrt{2}\left[\left(\frac{\pi}{2} + \frac{1}{2}\sin 2\frac{\pi}{2}\right) - \left(-\frac{\pi}{2} + \frac{1}{2}\sin 2\frac{-\pi}{2}\right)\right] = \sqrt{2}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \pi\sqrt{2}$$

Finally: $\boxed{P = 2P_1 = 2\pi\sqrt{2}}$

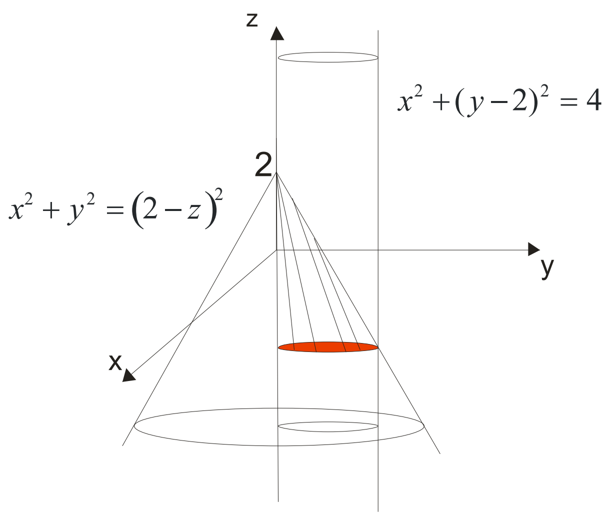
Example 3.

Calculate the surface area that cylinder $x^2+(y-2)^2=4$ cuts on the cone $z=2-\sqrt{x^2+y^2}$

Solution:

Let's get the cone to be able to draw a picture:

$$z=2-\sqrt{x^2+y^2}$$
$$\sqrt{x^2+y^2}=2-z$$
$$x^2+y^2=(2-z)^2$$



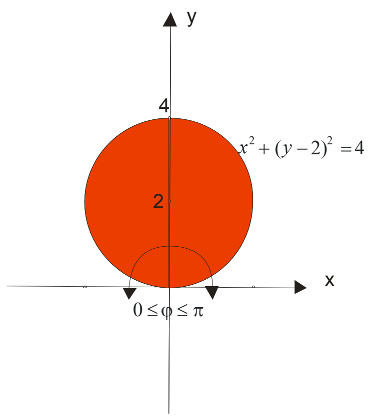
$$z=2-\sqrt{x^2+y^2}$$

$$p=\frac{\partial z}{\partial x}=\frac{-2x}{2\sqrt{x^2+y^2}}=\frac{-x}{\sqrt{x^2+y^2}}$$
$$q=\frac{\partial z}{\partial y}=\frac{-2y}{2\sqrt{x^2+y^2}}=\frac{-y}{\sqrt{x^2+y^2}}$$

$$\sqrt{1+p^2+q^2}=\sqrt{1+\left(\frac{-x}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{-y}{\sqrt{x^2+y^2}}\right)^2}=\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}=$$
$$=\sqrt{\frac{x^2+y^2}{x^2+y^2}+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}==\sqrt{\frac{2\left(x^2+y^2\right)}{x^2+y^2}}=\sqrt{2}$$

$$P = \iint_D \sqrt{1+p^2+q^2}\,dxdy = \iint_D \sqrt{2}\,dxdy = \sqrt{2}\boxed{\iint_D dxdy}$$

Surface area is actually area of the circle !



$$P_D = r^2\pi = 2^2\pi = 4\pi$$

$$P = \iint_D \sqrt{1+p^2+q^2}\,dxdy = \iint_D \sqrt{2}\,dxdy = \sqrt{2}\boxed{\iint_D dxdy} = 4\sqrt{2}\pi$$

$$\boxed{P = 4\sqrt{2}\pi}$$

Now let's look at previous task ... Well, there we could use this, right?

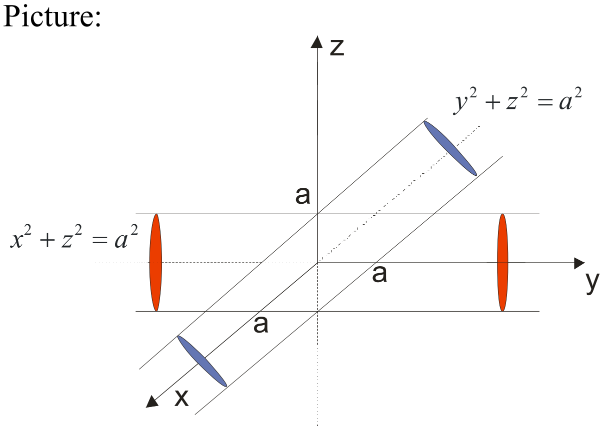
We have to show you both ways and you of course, do as your professor require!

Example 4.

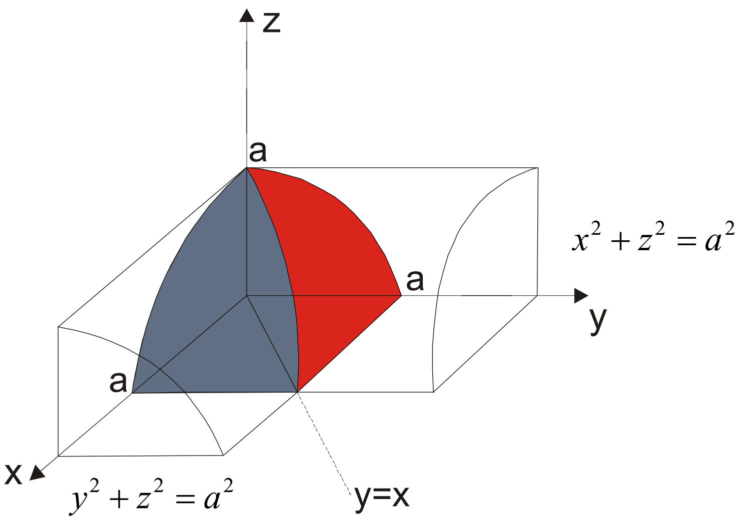
Calculate the surface area limited with $x^2+z^2=a^2$ and $y^2+z^2=a^2$.

<u>*Solution:*</u>

Picture:

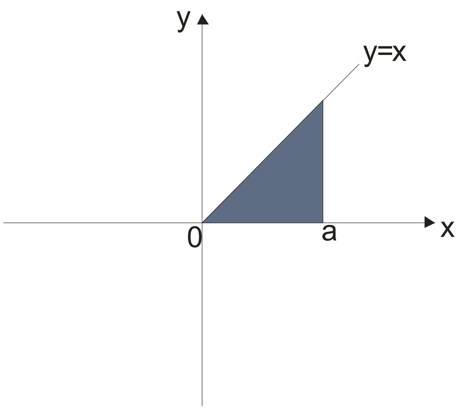


Now we separate the first octant.



Let us note here two surfaces. They are the same, and there are 16 counting all octant.

Find the limits in the plane xOy



$$D:\begin{cases}0\le x\le a\\ 0\le y\le x\end{cases}$$

$$x^2+z^2=a^2$$

$$z^2=a^2-x^2$$

$$z=\pm\sqrt{a^2-x^2}$$

We need first octant, so:

$$z=+\sqrt{a^2-x^2}$$

$$p=\frac{\partial z}{\partial x}=\frac{-2x}{2\sqrt{a^2-x^2}}=\frac{x}{\sqrt{a^2-x^2}}$$

$$q=\frac{\partial z}{\partial y}=0$$

$$\sqrt{1+p^2+q^2}=\sqrt{1+\left(\frac{x}{\sqrt{a^2-x^2}}\right)^2}=\sqrt{1+\frac{x^2}{a^2-x^2}}=\sqrt{\frac{a^2-x^2+x^2}{a^2-x^2}}=\sqrt{\frac{a^2}{a^2-x^2}}$$

$$\sqrt{1+p^2+q^2}=\frac{a}{\sqrt{a^2-x^2}}$$

$$P=16\cdot\int_0^a dx\int_0^x \frac{a}{\sqrt{a^2-x^2}}dy=16a\cdot\int_0^a \frac{dx}{\sqrt{a^2-x^2}}\int_0^x dy=16a\cdot\int_0^a \frac{xdx}{\sqrt{a^2-x^2}}$$

This integral we already solved:

$$\boxed{P=16a^2}$$

Example 5.

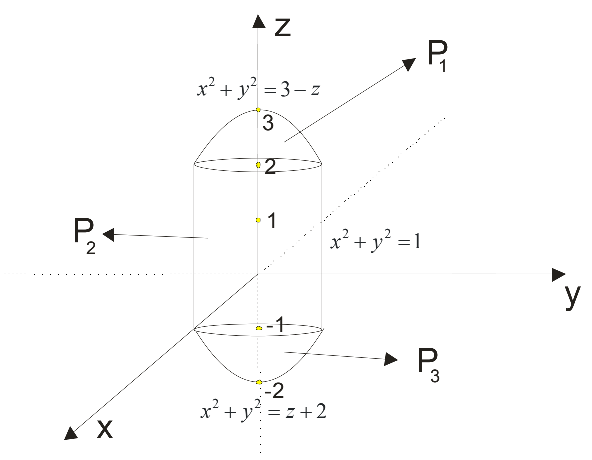
Calculate the area of the body which is limited with :

$x^2+y^2=z+2$

$x^2+y^2=1$

$x^2+y^2=3-z$

Solution:



We see that the desired area consists of three parts. Each of these areas will be found separately and then assemble the results ….

The first part of the surface is paraboloid $x^2+y^2=3-z$ which cuts off the cone $x^2+y^2=1$.

$$x^2+y^2=3-z$$
$$z=3-\left(x^2+y^2\right)$$
$$p=-2x$$
$$q=-2y$$

$$\sqrt{1+p^2+q^2}=\sqrt{1+4x^2+4y^2}=\sqrt{1+4\left(x^2+y^2\right)}$$

Cylinder $x^2+y^2=1$ determines the borders

$$\left.\begin{matrix} x=r\cos\varphi \\ y=r\sin\varphi \end{matrix}\right\} \rightarrow |J|=r \quad \text{then} \quad 0\le r\le 1 \quad \text{and} \quad 0\le\varphi\le 2\pi$$

$$P_1=\iint_D\sqrt{1+4\left(x^2+y^2\right)}dxdy=\underbrace{\boxed{\int_0^{2\pi}d\varphi}}_{\text{This is } 2\pi}\int_0^1\sqrt{1+4r^2}\cdot rdr=$$

$$\int\sqrt{1+4r^2}\cdot rdr=\left|\begin{matrix} 1+4r^2=t^2 \\ 8rdr=2tdt \\ rdr=\dfrac{tdt}{4}\end{matrix}\right|=\frac{1}{4}\int\sqrt{t^2}\cdot tdt=\frac{1}{4}\int t^2dt=\frac{1}{4}\frac{t^3}{3}=\frac{\left(\sqrt{1+4r^2}\right)^3}{12}$$

$$P_1=\iint_D\sqrt{1+4\left(x^2+y^2\right)}dxdy=\underbrace{\boxed{\int_0^{2\pi}d\varphi}}_{\text{This is } 2\pi}\int_0^1\sqrt{1+4r^2}\cdot rdr=2\pi\cdot\left.\frac{\left(\sqrt{1+4r^2}\right)^3}{12}\right|_0^1=$$

$$=\frac{\pi}{6}\cdot[\left(\sqrt{1+4}\right)^3-\left(\sqrt{1}\right)^3]=\frac{\pi}{6}\cdot\left(5\sqrt{5}-1\right)$$

$$\boxed{P_1=\frac{\pi}{6}\cdot\left(5\sqrt{5}-1\right)}$$

Next we will calculate the area P_3 , $x^2+y^2=z+2$ which cuts off the cone $x^2+y^2=1$

$$x^2+y^2=z+2$$
$$z=\left(x^2+y^2\right)-2$$
$$p=2x$$
$$q=2y$$

$$\sqrt{1+p^2+q^2}=\sqrt{1+4x^2+4y^2}=\sqrt{1+4\left(x^2+y^2\right)}$$

$$P_3=\iint_D \sqrt{1+4\left(x^2+y^2\right)}dxdy$$

So: $\boxed{P_3=\frac{\pi}{6}\cdot\left(5\sqrt{5}-1\right)}$ As we see this area is the same as the area in the previous section!

To find P_2. Let's look at the picture again. We see that this is actually Tread roller whose height is H = 3 and radius of base r = 1.

Why is H = 3 ?

If you solve the system $x^2+y^2=3-z$ and $x^2+y^2=1$ then z= 2

If you solve the system $x^2+y^2=z+2$ and $x^2+y^2=1$ then z = -1

Therefore, the height on the z- axis is 3.

$$P_2=2r\pi H$$
$$P_2=2\cdot 1\cdot \pi\cdot 3$$
$$\boxed{P_2=6\pi}$$

Now we add the results:

$$P=P_1+P_2+P_3$$
$$P=\frac{\pi}{6}\cdot\left(5\sqrt{5}-1\right)+6\pi+\frac{\pi}{6}\cdot\left(5\sqrt{5}-1\right)$$
$$P=\frac{\pi}{3}\cdot\left(5\sqrt{5}-1\right)+6\pi$$
$$\boxed{P=\frac{\pi}{3}\cdot\left(5\sqrt{5}+17\right)}$$